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(Formerly Mathematics News Letter)

VOL. 10 NOVEMBER, 1935 No. 2

In Appreciation

Administering Mathematics With Courage

Reflections of a College Teacher on Mathematics in the High School

Vector Analysis and Analytic Geometry

The Theory of Numbers for Undergraduates

The Teacher's Department

Notes and News

Problem Department

Book Reviews

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1. Through published standard papers on the culture aspects, humanism and history of mathematics to deepen and to widen public interest in its values.
2. To supply an additional medium for the publication of expository mathematical articles.
3. To promote more scientific methods of teaching mathematics.
4. To publish and to distribute to groups most interested high-class papers of research quality representing all mathematical fields.

In Appreciation

With the passing of the private status of this journal, goes the Editor's profound appreciation of the fine service rendered our cause by Inez (Mrs. E. J.) Land, who for the past several years has acted as Circulation Manager for the Magazine while holding the busy office of secretary of a progessive printing company, the Franklin Press.

Her contribution to the administration of this journal has not been the less valuable, or worthy of our gratitude, because of her extraordinary energy and capacity for work—even work over-time and on the side.

Her generous assistance cannot be forgotten.

S. T. SANDERS.

Administering Mathematics With Courage

With the possibility and even probability that required mathematics may be dropped from many high schools and even the lower grades, strong pressure should soon be brought to bear from proper sources. What seems to be behind it all is the fear to make the pupil think. The sugar coating of the pills is getting thicker continually, especially in mathematics and other difficult and fundamental subjects. We are now reaping the harvest of the system of "easing up". For, those who have been brought up under this pernicious system are now to too large an extent "teachers" in these subjects.

To my mind the solution of the baffling problem can be summed up in "bear down and insist on thoroughness". We need have no fear that the curricula of the schools will become too narrow. The urge to splurge is still with us. If mathematics teachers would "stick to their guns" and fail fearlessly those pupils who are really failing utterly we would doubtless have a new problem on our hands: a fight against flabby principals under the thumb of "know-nothing" boards of education, politically controlled superintendents and other influential enemies of true education.

H. B. CURTIS,
Lake Forest College, Illinois.

Reflections of a College Teacher on Mathematics in the High School

By RUDOLPH E. LANGER
University of Wisconsin

As I was browsing some time ago among my books, I found by chance in an ancient volume an allegory which I thought interesting, and when more recently I considered what I might say on this occasion, it occurred to me that with very few changes that legend might be so refashioned as to be not inappropriate to a gathering of teachers of the stern and precise doctrines of mathematics. It may be that you will find my resort to narrative contemptuous of your time and patience. Still, I am going to venture it, for after all there is a moral in the yarn if you but care to seek it. I have defaced the work it is true, but the original diction I have largely left, and the ancient author therefore speaks as follows:

"During one of those short furloughs which the soul may sometimes take, I found myself in a vast plain which I knew to be the valley of life. It possessed a great diversity of appearance, a sunny spot here and there a dark one, indeed just such a mixture of sunshine and shade as one may see on an April day when thin broken clouds are scattered over the heavens. In the very entrance to the valley there stood a huge and gloomy temple into which I seemed impelled to enter. It was a disfigured pile, covered with an excess of ornaments, many grotesque, many improbably fantastic, and many reminiscent of deformity in color and shape. Within, the temple was crowded with people, and though some danced with merriment in strange unintelligible figures, yet there was over all a palpable air of unquiet, many erred about in what seemed a pathetic puzzlement and perplexity, while others bore in their miens the marks of a melancholy or doubting resignation. Marshals in ceremonial robes seemed now to direct the movements of the throng, and again to drag with impatience after them some reluctant victim.

I stood the while lost in wonder: when lo! an attendant bade me uncover, for the place, he said, was the temple of a great idol. I bowed in awe before the priest, and he taking my offerings led me through dark and winding halls to a large and inner chamber. Not a lamp glimmered here, and yet, there was a half visibility which seemed to ensue from the wan and uncertain light of phosphoric rays issuing

from innumerable engravings upon the walls. The words of the inscriptions each by each methought I could read, but in sentences they were enigmas and riddles, totally incomprehensible.

The center of the chamber was filled with the great idol, ugly and hideous, and to look upon her filled me with an indefinable sense of uneasiness and trouble. Her features which rose out to view were ghastly and vague and uncertain, and her aspect of such pallor and so blended on the whole with darkness, that the beholder was denied at once every impression of kinship or distinctness. I prostrated myself and hastily withdrew, wondering and dissatisfied.

The main hall of the temple was filled now with a deep buzz as of many voices in discontent. The turmoil had yet increased, and as I stood uncertain there burst from the troubled throng an outcry, 'this is the temple of the idol of confusion, it is the house of misapprehension and error and bewilderment,' and therewith many rushed from the structure and I, methought, among them.

We sped from the temple with hasty steps, and having gone half the round of the valley were addressed by a woman, tall beyond the stature of mortals, with stately mien, and in her countenance that which though it showed reflection and sternness and sobriety of thought, yet was animated with an evident air of mastery and confidence. We inquired, and she replied: 'My name is Precision and Certitude. I am the spirit of Truth, and with me abides all that is orderly and unquestionable.'

Of our company some affrighted hurried onwards, but I and others, struck by the contrast of the form and manner of the spirit from that of the goddess of confusion, agreed to follow her. We ascended a great eminence which commanded the plain, and with the aid of the kindly spirit, we perceived the relations between the many parts of the land, of each to the other and of each to the whole. And lo! we could soon see with clearness even far beyond the limits of the valley of life.

And now with the quick transition of a dream I had rejoined those who had hurried on. Scorning to falter, and in eagerness to push onward, we soon had rounded the whole circle of the valley. And lo! there faced us the mouth of a vast cavern at the base of a lofty and perpendicular rock. Perplexed we entered the dusk of the cave. Two figures within guarded its mouth, and these were somehow revealed to me as Blunder and Prejudice. I recognized at once, that the great rock through which we had come, formed in fact only the extreme backward wall of the temple which we so assiduously had fled, and in the shock of the discovery I awoke."

I doubt not that you have been wishing the while that I might also awaken, that I might soon realize that the occasion before us is one for the discussion of serious questions, and not one for the mere frivolous telling of stories. There are serious questions, I know. With mathematics under attack, from sources, more than one, of pedagogic thought, there are indeed many serious questions. What mathematics shall the high school teach; to what students shall it be taught; and by what methods? These are questions which are currently much discussed. They are questions which it is important to discuss. I acknowledge this, and yet I am hoping still that you did not expect of me that I should be one to discuss them. For they, after all, it seems to me are, in a manner, questions peculiar to your household, and my knowledge of them must be of the less intimate kind, which is only proper for one who is but a neighbor next door.

Neighbors unfortunately are not always neighborly, and I fear that in the past that has been only too true of the members of your household and those of mine. We have, without enmity, to be sure, but also with little more than a formal cordiality been inclined to go our separate ways and in the matters of right along the line where our domains abut, we have generally chosen to insist rather jealously each upon his own prerogatives. This aloofness I think, and I am happy to say it, is rapidly waning. Among us in the colleges there are more and more who are awakening to a more immediate realization of the problems which you of the secondary school are facing. We have come to see a little more clearly that for us to ignore them is neither friendly nor wise, and you may count your allies among us in rapidly increasing numbers. This thawing-out of which I speak is no mere local manifestation, but a movement of truly wide scope. Hardly more than a month ago your National Council of Teachers of Mathematics met for the first time in conjunction with the American Mathematical Society and the Mathematical Association of America, and these three organizations thus initiated each other into a procedure of joint discussion of their problems.

I can see only profit for all concerned in such closer cooperation and understanding. We of the colleges, who are constantly engaged in the training of recruits for your very ranks, and you of the high-schools, who must do the initial shaping of the students who come to us, can neither of us afford to be indifferent or ignorant of the difficulties which the other faces, nor of the philosophies to which the other looks for light and guidance. I am not speaking now of your relations with college personnels in pedagogy, for with them your contacts have long been close. But of the mathematicians, though you are

teachers of mathematics, you have seen less; and yet, detached as they are from many of your more immediate perplexities, may it not be, that they may sometimes serve you by seeing more clearly than is possible for you the significance and implications of much that you do?

I cannot, at any rate, see how your counsels can lose by the addition of the voices of some whose outlook is that which comes from contact with the broader reaches of your subject, and from a familiarity with aspects of it which you do not ordinarily see. After all, your perplexities, while they are problems of teaching, are generally also problems of teaching mathematics. You have, no doubt, been warned of subject matter specialists, and the visionary reactions which they are popularly supposed to bring to concrete situations and needs. But these same specialists are on the whole constantly meeting teaching problems of their own, and somehow they do seem in the managing of them to find it possible to be at least moderately practical. A former college teacher was once nominated for the Presidency of the United States, and proved to be so practical a campaigner, that he was too much for his political opponents, and that despite the fact that they lost no conceivable opportunity of referring to his as "the professor."

To justify the teaching of mathematics it was once necessary only to point to the great age and venerability of the subject, how it had been studied in the hey-day of Greek culture, and by the peoples that came thereafter, and even by the Egyptians before. But our time now is one of iconoclasm, and venerability itself is no longer a buttress, often enough indeed almost the contrary. Our schoolmen, at least many of them, have become disdainful of tradition, and some, who have allowed their enthusiasms to outrun themselves, sometimes seem to see a subject as good only if it is new, and to find in the harboring of any other, the clearest evidence of an indifference to the advance of educational thought. Now to be sure old subjects have often and rightly had to yield their places to new ones, and mathematics is undeniably old. There is, however, a distinction not to be slighted between mathematics and almost all subjects which in the past have fallen into discard, and that is this, that despite its age mathematics shows no slightest sign of either obsolescence or stagnation.

If one passes in review the gamut of subjects in which men have in times been interested, it may be observed that in many of these it was, or even is, customary for each generation to demolish and discard what the previous one had built. Other subjects again advanced and grew only to attain a state of completeness, perhaps even one of perfection, for which the only sequel could be one of decadence and antiqua-

tion. Now from each of these categories mathematics is notably absent. In this subject men of all times have worked in harmony, and each generation has forged its accomplishments only from the metal and steel which its forerunners had prepared. The development of mathematics in this respect resembles nothing so much as the development of the whole of human civilization, for neither would ever have been possible had not men of every time accepted as an inheritance the accumulated achievements and experiences of the past, always for the production of yet greater things. The capacity to do this, more than any other, is one which distinguishes man, and the great structure of mathematical theories is thus eminently a monument to human genius. The foundations of this monument were to be sure laid in antiquity, and stone has been added to stone as the ages have gone by. Yet no greater error could one make than to regard it now as a finished relic, for in the present more than ever in any time past does it find itself in the throes of construction and growth. Mathematics is old, but also it is as new as the newest, for of all subjects none other so combines the elements of permanence and stability with an unmeasurable capacity for growth and change.

Carlyle on an occasion characterized the teachers of his time by saying of them that they knew of the human soul only this: that it had a faculty called memory which could be acted on by birch rods. We pride ourselves that teachers have come far since days like those, but let us remember that in education as in most things there is no alternative to progress but that of regression. The profession advances or it recedes, and it does the former only under continual stimulation. It behooves all of us, therefore, to be interested in improvement of of our work, to give an intelligent ear to every new thought, and to avail ourselves of every real gain. If mathematics is to remain a subject continually proper for schools which are abreast of the times, it and its pedagogy, like any other subject, must submit to, and remain in, a state of continual readjustment. The search for improvements must be a constant one, and every teacher must seek to make each new find his own. By so doing criticism will be largely forestalled, and by so doing only may we assure ourselves of the power at all times to defend our positions. Meanwhile our past need not shun scrutiny. There have been advances, many and significant ones, in the teaching of our subject, both in your field and in mine. To be sure these advances have not been revolutionary enough for some. But our subject after all is no fly-by-night, and our contemporaries are far from being the first that have devoted serious thought to it.

It is a criticism of mathematics, in the minds of some, that as a subject it is hard. One's philosophy of life must be a flabby one who can see that only as a disparagement. I, for one, prefer much to array myself with our great public figure Charles Evans Hughes, who, speaking before the National Education Association, felt himself moved to say:

"As I look back upon my own experience I find that the best lessons of life were the hardest. . . . Life is not a pastime and democracy is not a holiday excursion. It needs men trained to think, whose mental muscles are hard with toil and who know how to analyze and discriminate. . . . I am," he says, "one of those who believe in the classical and mathematical training, and I do not think we have found any satisfactory substitute for it. The true object of education is that sort of training which insists, at whatever cost, on accuracy—the lack of which, I regret to say, is now conspicuous in students of all grades. There is at present a bewildering and unsuccessful attempt at comprehensiveness. It fails of its purpose in giving neither adequate information nor discipline . . . ."

Of course it is not necessary to advocate at the other extreme, as was once no uncommon practice, that mathematics must be good *because it is hard*. Happily enough, we teachers of the subject are far from being reduced to a resort to any such paltry claim. Yet I find little to be said for those who think that pupils should never have to do but what they like to do, or that a subject to be a proper one must always be either interesting or exciting. It was with masterly derision of such a theory that Dean Briggs of Harvard said once:

"We wabble all over the flowery meadows—(until) the student virtually comes to the teacher with a bill of rights in his hands and says: "Mind you must attract me or I shalln't get on, you must rivet my attention for otherwise my thoughts will wander, and you must not be dull or I will surely go to sleep." "Soon," he says, "students will become so reduced that they will wish, as it were, to lie in bed and have their studies brought up to them."

We have all and many times heard it voiced that mathematics as commonly taught is barren and dry, that it is devoid of inspiration, that the epithet "formula grinding" aptly characterizes it. Let us face this issue squarely; in the hands of uninspired teachers, or of ignorant or careless ones, our subject undoubtedly has the greatest of potentialities for degeneration into monotonous and deadly routines. On this score our critics are surely not always wrong. May we not do best to admit the fault, and having done so, face the challenge to be overcome. Now no teacher for whom personally mathematics

is dead and dry can teach it to others as a vital living subject. To keep it a living subject one must continue to educate one's self concerning it. The intellectual equipment unless kept in a constant state of growth invariably ossifies, and as its resiliency wanes the power for real teaching wanes with it.

To be sure we are often enough sinned against by being charged with faults we may disown. For mathematics is all too often taught, and through no fault of ours, by teachers who have neither training for the task nor a glimmer of comprehension of the real significance of the subject. To such teachers its historical significance is a closed book; rigor and abstraction are for them terms they do not understand; they have little or no conception of the roles of inductive and deductive reasoning; and the power and subtlety and refinement of an algebraic symbolism is utterly beyond the realm of their appreciations. How can mathematics be other than a dry and barren subject in such hands, when true teaching is always a serious challenge even to the best trained among us?

"In poetry," says Goethe, "there are two kinds of dillentanti; those who neglect the indispensable mechanical part, and think they have done enough if they show spirituality and feeling; and those who seek to arrive at poetry without soul or matter merely by mechanism." If one strikes from this statement the term "poetic dillentanti" and substitutes for it "uninspired teachers of mathematics," not the slightest further change in either word or thought need be made. They are poor and desultory teachers that glorify the mechanism to the detriment of thought. Their pupils learn mechanically, and carry away through life a lasting aversion and antagonism for the subject. They are incredulous if ever they hear that mathematics goes hand in hand with culture, that it embodies many of the profoundest of human thoughts, and of the most heroic conquests of the imagination. Can one blame them, if they forever scoff at the notion that mathematics is filled with beauty, or that a great mathematician must have a superlatively fine artistic sense?

But they, on the other hand, are likewise poor teachers who neglect any reasonable development of technique. The individual from childhood on can acquire power over number relations only by contact and by doing. No one has ever yet mastered mathematics by the mere hearing about it, and that the true teacher knows. He appreciates the thought and significance of his subject, and at the same time is aware of the extent to which technique is indispensable and yet adequate. Such teachers may scorn the subterfuge of masking their subject under an artificial veneer of utilitarian terms in order to

fortify its place in the curriculum. For them the subject lives and justifies itself in the classroom.

Are we in defending mathematics perhaps after all fighting for a forlorn cause? In the face of criticism some of you perhaps have been doubtful, and have wondered whether in the distant, if not in the near future, the pressure of other subjects will not force mathematics out of the high-school curriculum. I think not: for I do not believe that our schools can ever afford to deny instruction in any subject which is vital to a real understanding of our culture in its broadest sense, or which promises, as does mathematics, to become more and more the essence of all precise knowledge. I am not overdrawing the picture. Sir James Jeans, a great scientist, wrote only recently in Nature saying:

"Our knowledge of the world must always consist of numbers, and our picture of the universe—the synthesis of our knowledge— must necessarily be mathematical in form. All the concrete details of the picture—are mere clothing that we ourselves drape over our mathematical symbols. They belong to the parables by which we try to make nature comprehensible."

This is not a surprising statement. The trend of knowledge toward standards of accuracy and precision and quantitative analysis is an acknowledged fact in almost every field of learning. Within the last decade there was founded in this country a so-called Institute of Advanced Study. It is a sort of super-university to which only those go to study who have already attained the degree of Doctor of Philosophy. This institute at present teaches only mathematics. Why mathematics? The official reply made at its opening was the following: "After consultations with scholars throughout the world it was decided to start with a school of mathematics as the most fundamental of all the disciplines."

I could not urge you to make the significance of mathematics known to your students. That task would be beyond your human powers. But to do what can be done along this line is manifestly desirable, and for you, here in Wisconsin, that is the more so now than it has been heretofore, because of the recent developments in the relations between your secondary schools and our university. I am referring to the new entrance requirements of the university, which have come into operation in the current year. Though I believe you are all aware of some of the changes which were made, I would nevertheless like to dwell briefly here upon them.

To begin with mathematics is no longer an absolute requirement for admission. When baldly stated so the fact certainly sounds quite

the opposite of an endorsement of the importance of the subject. To see it as such, however, is to fundamentally misunderstand the truth of the matter. With the purpose of liberalizing its educational policies to accord with the best of modern thought on these subjects, the university has virtually agreed to say to the student who in the future hopes to seek admission: if you have no liking for mathematics and no ability to pass in mathematics courses, and if you will wish to carry on your major studies only in the fields of languages and history and journalism or a few other special subjects, we shall grant you admission. However, your standing in the university will be "restricted." You will not be free to change your mind after you are here; and if you find other fields of study more attractive you will also find that major study in them is closed to you, for that requires "unrestricted standing." Such standing is granted only to those who pass the courses in algebra and plane geometry, or who by subsequent private study make up the deficiency. As a "restricted" student you will be barred from majoring in any science, be it chemistry or physics or biology, or astronomy or geography, or psychology or pharmacy or medicine. You will be barred from majoring in philosophy, for philosophy and mathematics are in many ways notably akin, and the role of mathematics in the social sciences is now of such importance that you will be barred from major work in political science or sociology, as well as in economics or commerce. The School of Engineering will, of course, be closed to you, and so also will be the College of Agriculture.

Such, in brief, are the new university regulations. You will find them printed in detail in the forthcoming university catalogue. To the student without mathematics the university is open, but only in part, and this is your duty to make generally understood. Few students know when they begin the high school what their interests in the university will be, and you will not be blameless if they learn too late the penalties of shunning your subjects.

Science is best expressed when it is mathematically expressed; and yet most students of natural science cringe at the prospect of higher mathematics. It is because the author has seen so many students struggling against the handicap of an inadequate preparation in mathematics—spending more effort in avoiding mathematics than needs to be spent in learning it—that this book has been written.—Farrington Daniels in *"Mathematical Preparation for Physical Chemistry."*

Vector Analysis and Analytic Geometry

By WILLIAM E. BYRNE

Two vector concepts occur in many problems of analytic geometry, the conditions of perpendicularity and of parallelism. After obtaining an analytic expression of the conditions several elementary applications will be made. The notation * indicates an immediate generalization from two to three dimensions; unless otherwise indicated the discussion will be for the case of two dimensions.

**Scalar product of two vectors.* Given the two vectors of components (X_1, Y_1) and (X_2, Y_2). If $\vec{i}$ and $\vec{j}$ represent unit vectors along Ox and Oy (rectangular coordinates), respectively, the given vectors may be represented by the equivalent vectors

$$\vec{V}_1 = \vec{OP}_1 = X_1\vec{i} + Y_1\vec{j}$$
$$\vec{V}_2 = \vec{OP}_2 = X_2\vec{i} + Y_2\vec{j}$$

The scalar product of two vectors $\vec{V}_1, \vec{V}_2$ is defined as the product of their lengths and of the cosine of the angle between them. It is denoted by $\vec{V}_1 . \vec{V}_2$, and it is often called the "dot" product.

If we apply the law of cosines to the triangle OP_1P_2 (Fig. 1) we find

$$OP_1^2 + OP_2^2 - 2OP_1 . OP_2 \cos\theta = P_1P_2^2$$

or
$$X_1^2 + Y_1^2 + X_2^2 + Y_2^2 - 2\vec{V}_1 . \vec{V}_2 = (X_2 - X_1)^2 + (Y_2 - Y_1)^2$$

which gives

$$\vec{V}_1 . \vec{V}_2 = X_1X_2 + Y_1Y_2 \tag{1}$$

The corresponding formula for three-space is

$$\vec{V}_1 . \vec{V}_2 = X_1X_2 + Y_1Y_2 + Z_1Z_2 \tag{2}$$

Hence the condition that two vectors be perpendicular is that their scalar product be zero. In particular a null vector, i. e. a vector of components (0,0), is considered to be perpendicular to all vectors, including itself.

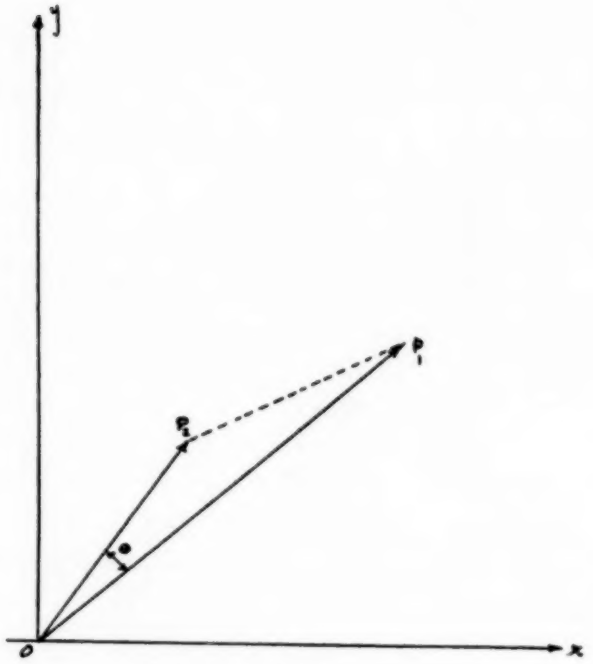

Fig. 1

**Parallel vectors.* Two vectors are parallel if, and only if, there there exists a relation

$$\vec{V}_1 = \lambda \vec{V}_2 \tag{3}$$

(V_2 being a non-null vector). In terms of components (3) is equivalent to

$$\begin{cases} X_1 = \lambda X_2 \\ Y_1 = \lambda Y_2 \end{cases}$$

**Application* 1. Equation of a straight line, parametric form.

Let $P_0(x_0, y_0)$ be a point on the line L and (A,B) be the components of a vector $\vec{U}$ parallel to the line L.

Let M(x,y) be any point on L. Then

$$\vec{P_0M} = \lambda U \tag{4}$$

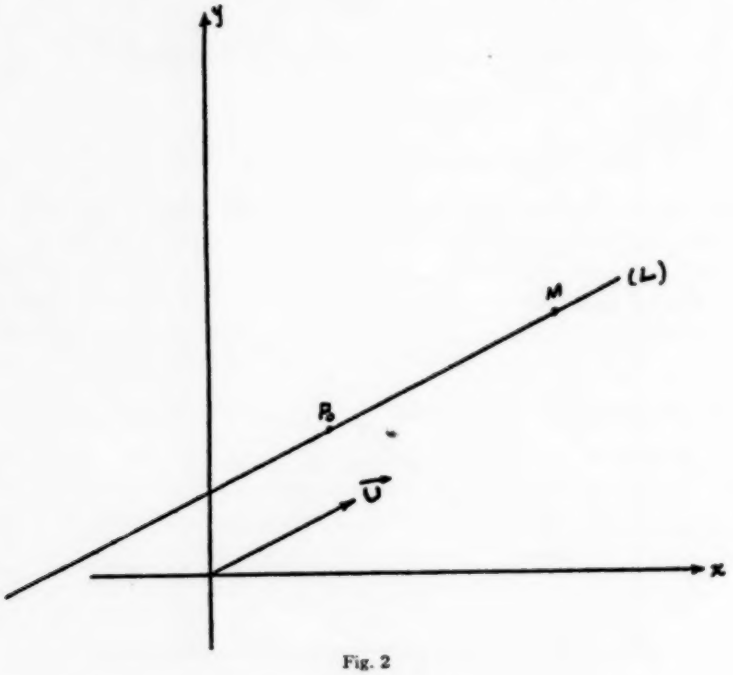

Fig. 2

by (3). The corresponding relations between the coordinates may be written

$$\begin{cases} x-x_0=\lambda A \\ y-y_0=\lambda B \end{cases}$$

As λ varies from $-\infty$ to $+\infty$ M traces out the entire line L. The components (A,B) of U are called the *direction numbers* of L; if $\vec{U}$ is a unit vector, i. e. if $A^2+B^2=1$, (A,B) are called the direction cosines of L. There are two sets of direction cosines for a non-oriented line, but only one set of direction cosines for an oriented line.

Application 2. Interpretation of the coefficients A, B in the general equation of the line L, $Ax+By+C=0$.

Let (A, B) be a vector $\vec{V}$. Let (x_0,y_0) be the coordinates of any point on L. Then

$$Ax_0+By_0+C=0$$

If this be subtracted from the given equation we have

$$A(x-x_0)+B(y-y_0)=0 \tag{5}$$

which by (1) shows that $\vec{V}$ is perpendicular to $\vec{P_0M}$ (see figure 3).

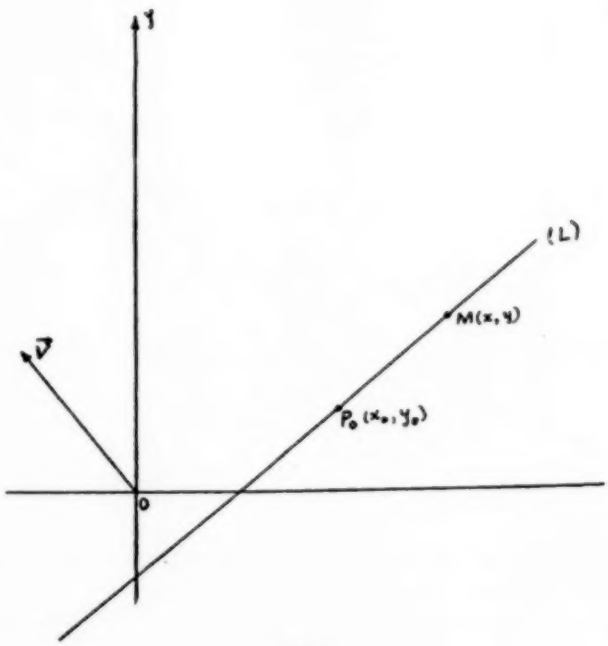

Fig. 3

As a corollary of this we may find the angle between two lines:

$$(L_1) \qquad A_1x+B_1y+C_1=0$$

$$(L_2) \qquad A_2x+B_2y+C_2=0$$

$$\vec{V}_1 \cdot \vec{V}_2 = A_1A_2+B_1B_2 = \sqrt{A_1^2+B_1^2}\ \sqrt{A_2^2+B_2^2}\cos\theta$$

$$\cos\theta = \frac{A_1A_2+B_1B_2}{\sqrt{A_1^2+B_1^2}\ \sqrt{A_2^2+B_2^2}} \tag{6}$$

This formula does not have the disadvantage of the use of slopes for there are here no exceptional lines devoid of slope (lines parallel to the y axis).

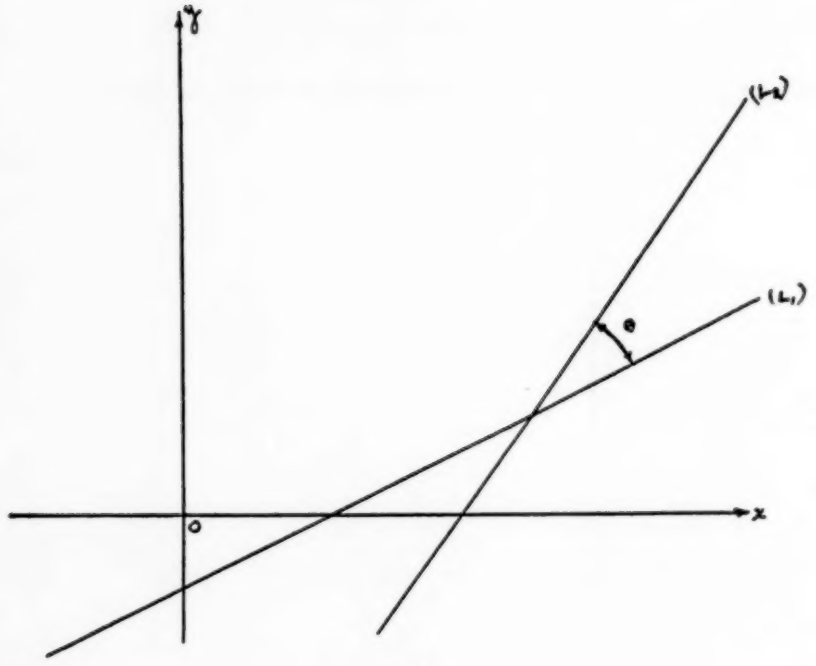

Fig. 4

The generalization of this type of equation to three dimensions gives the angle between two planes P_1 and P_2.

$$(P_1) \qquad A_1x+B_1y+C_1z+D_1=0$$

$$(P_2) \qquad A_2x+B_2y+C_2z+D_2=0$$

$$(7) \qquad \cos\Theta=\frac{A_1A_2+B_1B_2+C_1C_2}{\sqrt{A_1^2+B_1^2+C_1^2}\ \sqrt{A_2^2+B_2^2+C_2^2}}$$

Formula (7) also gives the angle between two lines in three dimensions of equations

$$(L_1) \qquad \begin{cases} x=x_1+\lambda A_1 \\ y=y_1+\lambda B_1 \\ z=z_1+\lambda C_1 \end{cases} \qquad (8)$$

$$(L_2) \qquad \begin{cases} x=x_2+\mu A_2 \\ y=y_2+\mu\, B_2 \\ z=z_2+\mu\, C_2 \end{cases} \qquad (9)$$

Application 3. Normal equation of a straight line.

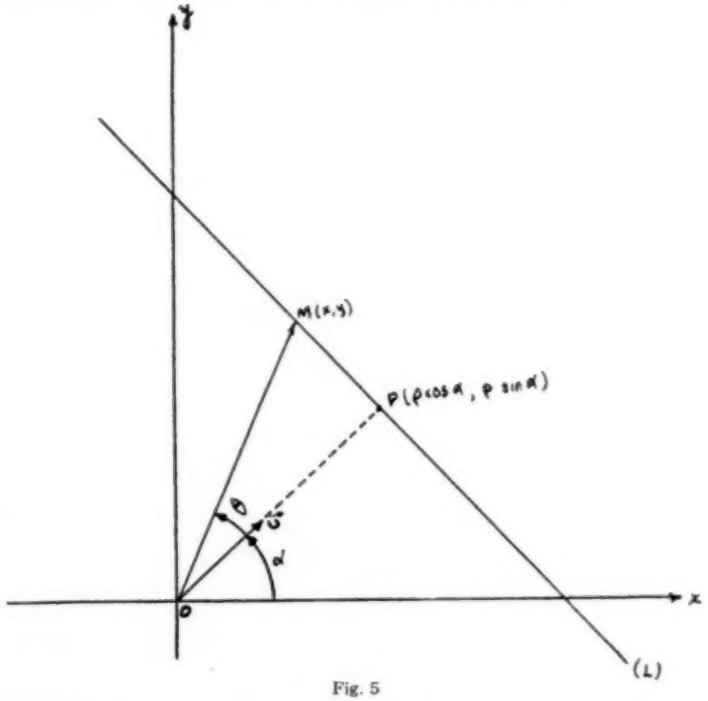

Fig. 5

A straight line L may be located by means of the polar coordinates of the foot of the perpendicular P on L. Let these coordinates be (ρ, α). For the figure indicated $\rho > 0$, but α may be replaced by $\alpha + \pi$ and ρ by $-\rho$ without changing the validity of the argument.

$$\overrightarrow{OM} = x\mathbf{i} + y\mathbf{j}$$

$$\overrightarrow{U} = \cos\alpha . \mathbf{i} + \sin\alpha . \mathbf{j}$$

$$\overrightarrow{OP} = \rho\cos\alpha . \mathbf{i} + \rho\sin\alpha . \mathbf{j}$$

$$\overrightarrow{OM} . \overrightarrow{U} = x\cos\alpha + y\sin\alpha = (\overrightarrow{OP} + \overrightarrow{PM}) . \overrightarrow{U} = \rho$$

Hence the equation of L is

$$x\cos\alpha + y\sin\alpha = \rho \tag{10}$$

In the general equation of a straight line L

$$Ax + By + C = 0$$

$\overrightarrow{V} = A\overrightarrow{i} + B\overrightarrow{j}$ is a vector perpendicular to L, hence

$$A = \lambda \cos \alpha$$

$$B = \lambda \sin \alpha$$

$$A^2 + B^2 = \lambda^2$$

$$\lambda = \epsilon \sqrt{A^2+B^2} \quad (\epsilon = +1 \text{ or } -1)$$

and

$$(11) \qquad \begin{cases} \cos \epsilon = \dfrac{\epsilon A}{\sqrt{A^2+B^2}} \\ \sin \alpha = \dfrac{\epsilon B}{\sqrt{A^2+B^2}} \\ \rho = \dfrac{-\epsilon C}{\sqrt{A^2+B^2}} \end{cases}$$

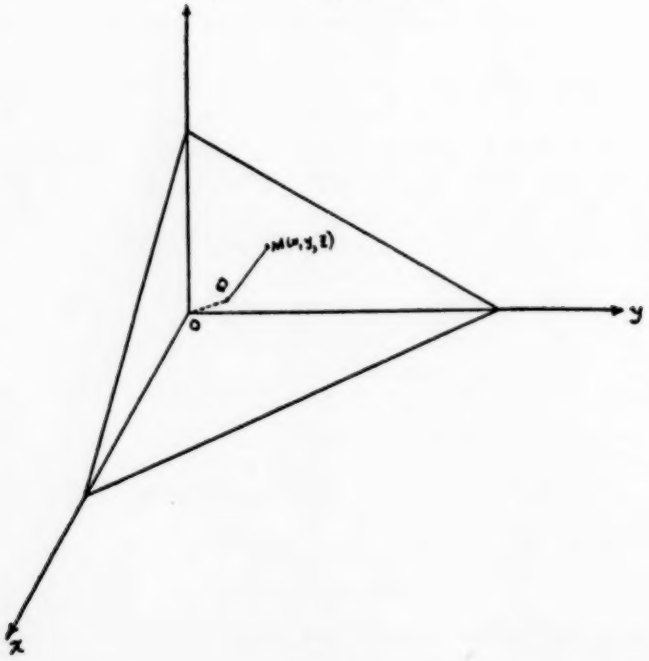

Fig. 6

There is a corresponding problem in three dimensions to find the normal equation of a plane Q. Let (α,β,γ) be the direction angles of a perpendicular OP from O to the plane.

$$\vec{U} = \cos\alpha.\vec{i} + \cos\beta.\vec{j} + \cos\gamma.\vec{k}$$

$$\vec{OQ} = \rho\vec{U}$$

$$\vec{OM}.\vec{U} = x\cos\alpha + y\cos\beta + z\cos\gamma = (\vec{OQ} + \vec{QM}).\vec{U} = \rho. \quad \text{Hence}$$

$$x\cos\alpha + y\cos\beta + z\cos\gamma = \rho \tag{12}$$

Application 4. Perpendicular distance between two skew lines.

Consider the lines (8), (9). Let $\vec{U}$ be a unit vector perpendicular to both lines.

$$\vec{U} = \cos\alpha.\vec{i} + \cos\beta.\vec{i} + \cos\gamma.\vec{k}$$

The perpendicularity is expressed by the system of equations

$$\begin{cases} A_1\cos\alpha + B_1\cos\beta + C_1\cos\gamma = 0 \\ A_2\cos\alpha + B_2\cos\beta + C_2\cos\gamma = 0 \end{cases} \tag{13}$$

which admit a single infinity of solutions if the matrix

$$\begin{Vmatrix} A_1 & B_1 & C_1 \\ A_2 & B_2 & C_2 \end{Vmatrix}$$

is of rank 2. Of all these solutions only two sets (corresponding to the two possible orientations) fulfill the condition

$$\cos^2\alpha + \cos^2\beta + \cos^2\gamma = 1 \tag{14}$$

Geometrically (13) represents a line L through 0 and (14) a sphere S of center 0 and radius 1 if $\cos\alpha$, $\cos\beta$, $\cos\gamma$ are considered as the coordinates of a point. The required solutions correspond to the two points in which L and S intersect. The vector

$$(x_2 - x_1)\vec{i} + (y_2 - y_1)\vec{j} + (z_2 - z_1)\vec{k}$$

joins a point P_1 of L_1 to a point P_2 of L_2. Let Q_1, Q_2 be the feet of the common perpendicular on L_1 and L_2. We have

$$\vec{P_1P_2}.\vec{U} = (\vec{P_1Q_1} + \vec{Q_1Q_2} + \vec{Q_2P_2}).\vec{U}$$

$$= \vec{Q_1Q_2}.\vec{U}$$

Hence $$= (x_2 - x_1)\cos\alpha + (y_2 - y_1)\cos\beta + (z_2 - z_1)\cos\gamma$$

$$(15) \quad Q_1 Q_2 = |(x_2 - x_1)\cos\alpha + (y_2 - y_1)\cos\beta + (z_2 - z_1)\cos\gamma|$$

Only a few of the many possible applications of vector concepts to analytic geometry have been given here. The situation becomes somewhat more complicated when oblique coordinates are used, but the fundamental concepts remain unchanged except for the introduction of two systems of components or coordinates of a vector, which transform differently under general linear transformations of coordinates.

A perfect science is a dead science. Happily for those whose chief interest is mathematics, this most abstract of all sciences is far from perfect, and the rate at which it is growing today indicates that a perfect mathematics is not to be expected before the human race itself attains perfection.—Eric T. Bell in April, 1934, issue of *"Current History."*

It is common knowledge that solid geometry is less widely taught than formerly. That is, a smaller proportion of students completing plane geometry go on to solid geometry than heretofore. This may simply be the result of the students' preference, freely expressed under elective systems of one sort or another. Even a teacher of mathematics, while not discounting the value of the three dimensional experience, might well question the desirability of devoting a year and a half to demonstrative geometry in the secondary school. No one knows the relative value of eight months of geometry as compared with five; nor is there any information available as to just how many months the student should demonstrate and argue about geometric abstractions before he attains in reasonable measure to those disciplinary and cultural aims which we like to think our teaching of geometry achieves. There does seem to be widespread agreement, however, that the student should not bring his geometric experience to a close without having considered the three dimensional aspects of the world about him. Even before the recent decline in the popularity of solid geometry, our instruction in the mathematics of three dimensions—in colleges as well as in secondary schools—suffered in comparison with that of European schools. We do not need to remain forever complacent about this unfavorable comparison, simply because it is now so familiar to us.—Ralph Beatley in *American Mathematical Monthly*, February, 1928.

The Theory of Numbers for Undergraduates

By EMORY P. STARKE
Rutgers University

"The elementary theory of numbers should be one of the very best subjects for early mathematical instruction. It demands very little previous knowledge; its subject matter is tangible and familiar; the processes of reasoning which it employs are simple, general and few; and it is unique among the mathematical sciences in its appeal to natural human curiosity. A month's intelligent instruction in the theory of numbers ought to be twice as instructive, twice as useful, and at least ten times as entertaining as the same amount of 'calculus for engineers'." "It is unique among mathematical theories in its appeal to the uninstructed imagination and in its fascination for the amateur."

The quotations are from a lecture* given by Prof. G. H. Hardy during his last visit to America. The professor may be a bit hard on calculus, but his advice that the theory of numbers will prove a profitable addition to the undergraduate curriculum is worthy of consideration.

The writer has been privileged to give such a course for a number of years at Rutgers University,—with highly satisfactory results as regards both student interest and mathematical growth. In few other mathematics classes does it become necessary to restrain the eagerness of students in seeking additional problems† and attempting their solutions. We are particularly desirous that prospective teachers of mathematics shall take the course, since we believe that a teacher fascinated with his subject is the best teacher.

The one drawback in adapting number theory to undergraduate students is the lack of a suitable text. There seems to be nothing in English more adequate than the monograph without being too difficult or too elaborate for our use. Fortunately there are few subjects in which, without a text, material can so easily be organized for class use. An enterprising instructor, himself interested in the work, will find little difficulty in putting together an attractive and valuable course of study. He will want for reference a few of the standard

*The sixth Josiah Willard Gibbs lecture, given in New York City December 28, 1928. See the Bulletin of the Am. Math. Soc., v. 35 (1929), pp. 778-818.

†Thirty-eight of the problems printed or solved in the first six issues of the *American Mathematical Monthly* for 1934 involved elementary number theory.

works,* and will perhaps put into the hands of his students a monograph like Carmichael's, "Theory of Numbers" (Wiley). If Dickson's "History of the Theory of Numbers", at least vol. I, is available, it will prove an invaluable source of information on many suggested problems.

The following syllabus, approximately that used at Rutgers, is suggested as representing a fairly adequate course of study for one semester. The length of time devoted to each topic will depend at the discretion of the instructor upon the ability and interests of the class. For undergraduate instruction, an adequate theory must be supplemented with a wealth of problem and exercise material. This can be culled from many sources, although much of it will be constructed by the instructor to fit his particular need. The class should be trained to develop ingenuity in attacking new problems—an unpromising point of view frequently yields an easy solution after more obvious ideas have failed. Hence the student must be ready to view each problem from many angles and not always expect his first attack to be successful.

I. Divisibility.
   1. Fundamental notions and laws.
      a) The arithmetic of positive integers assumed.†
      b) Mathematical induction.
   2. Divisors.
      a) Definitions of divisor, multiple, prime, relatively prime, etc.
      b) If $b|a$ ‡ and $c|b$, then $c|a$.
      c) If $c|a$ and $c|b$, then $c|(a \pm b)$.
      d) Every integer greater than 1 has a prime factor.
      e) Given a and b, there exist integers q and r such that $a = qb + r$, $q \geqq 0$, $0 \leqq r < b$.
   3. The greatest common divisor.
      a) The Euclidean algorithm.
      b) If p is prime and $p|ab$, then either $p|a$ or $p|b$.
      c) The unique factorization theorem.‡
      d) If d is the greatest common divisor of m and n, there exist integers x and y such that $mx - ny = \pm d$.

*A short bibliography is given at the end of this paper.

†I have for some classes found it worth while to outline the logical derivation (from undefined terms and postulates) of the fundamental laws of the arithmetic of integers.

‡$b|a$ is the symbol for "b is a divisor of a".

‡Emphasize the significance of this theorem by constructing an "arithmetic" in which it does not hold. See p. 813 of Hardy's article.

e) Solution of indeterminate equations of the form $ax \pm by = c$.
f) The least common multiple.

4. Formulas concerning divisors.
   a) The sum of the divisors of an integer.
   b) The number of divisors of an integer.
   c) Perfect numbers.*
   d) Amicable numbers.
5. Additional topics.
   a) Given n and p (prime), find greatest x such that $p^x | n!$.
   b) If $n = a + b + \ldots + k$, then $n!/a!b!\ldots k!$ is an integer.
   c) There are infinitely many primes.
   d) Scales of notation.†
   e) Mention of more advanced results about primes, and of some conjectures as yet unproved.

II. Congruences.
1. Definitions and elementary manipulations.
2. Linear congruences.
   a) Number of solutions.
   b) Method of solution, analogous to I.3.e).
   c) If m and n are relatively prime, there exist integers x for which simultaneously $x \equiv a \pmod m$ and $x \equiv b \pmod n$. All such values of x are congruent to each other (mod mn).
3. A congruence of degree n with a prime modulus has not more than n roots.

III. Fermat's theorem and applications.
1. Euler's totient, $\phi(a)$.
   a) Method of calculation.
   b) The sum of the totients of the divisors of n is n.
2. Euler's generalization of the Fermat theorem.
   a) Proof and illustrations.
   b) An additional hypothesis needed for its converse.
3. Wilson's theorem and converse.
4. Cauchy's maximum indicator.‡

*Perhaps also 'multiply-perfect numbers". See Carmichael: Bulletin of the Am. Math. Soc., v. 13 (1907), p. 383.

†Much profitable amusement arises when the class first attempts to perform additions, divisions, etc. with numbers expressed in a scale different from 10.

‡Called $\lambda(a)$ by Carmichael.

5. Quadratic congruences.
   a) Euler's criterion for a quadratic residue to a prime modulus.*
   b) Determination of quadratic residues for small moduli.
   c) Solution of numerical quadratic congruences.†

IV. Exponents.
1. Exponent of an integer to a modulus.
   a) Definition and illustration.
   b) If d is the exponent of a (mod m) and if $a^x \equiv 1$ (mod m), then d|x.
   c) If d is the exponent of a (mod m), then $d|\phi(m)$.
2. Primitive roots (mod m).
   a) Proof of existence for $m = 2^c p^d$, c = 0 or 1, p a prime.
   b) If $x|\phi(m)$, then there are $\phi(x)$ integers of which x is the exponent.
   c) If a is a primitive root (mod m), every integer prime to m is congruent to just one of the integers, $a, a^2, \ldots, a^{\Phi(m)} \equiv 1$.
   d) If a is the exponent of x (mod m), and if g is the quotient of a and the highest common divisor of d and a, then g is the exponent of x^d (mod m).
   e) If m and n are relatively prime, and a and b are the exponents of x to the moduli m and n respectively, then the least common multiple of a and b is the exponent of x (mod mn).
3. Indices (prime modulus).
   a) Theorems analogous to logarithms.
   b) Construction of a table for a small modulus.
   c) Solution of congruences of type $ax^n \equiv c$ (mod p).

V. Diophantine equations.
1. Review of I.3.e).
2. Indeterminate systems of linear equations.
3. Quadratic Diophantine equations.
   a) Solution of $x^2+y^2=z^2$.
   b) Simplest cases of $ax^2+bxy+cy^2=ez^2$.
4. Mention of Fermat's last theorem.

*Some reference to the Legendre symbol and the reciprocity law is well worth while.

†No all-inclusive rule for this can be given. With completing of the square and the use of III . 5 . b), each problem should suggest its own procedure.

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There are very few freshmen who are adequately prepared for and who can successfully master college algebra, trigonometry, and analytical geometry in one year, four periods per week. It requires time to acquire mathematical ideas and much training for mastery. To be frank we shall have to admit that college algebra is a conglomeration of topics. We admit that they are needed and badly needed; but they are not built around a central idea. I refer to the rational functions. In trigonometry we give the student a smattering of the circular functions, but we do not let him hear of their inverses. Analytical geometry offers the first opportunity to the student to use all his mathematical knowledge. It is the proving ground. The average student has just about enough time to learn the terminology connected with the course, but not enough of the technique required for enjoyment. He learns that there are such things as the straight line and the standard forms of the conics. He is allowed a glimpse of polar coordinates, higher plane curves, and transformation of coordinates. His solid analytical geometry is sealed with a great seal. He finished his course with no idea of what the fundamental problem of analytical geometry is. He has graphed a few curves but he has not been given time to study the graphs after they have been made.—Roscoe Woods in *American Mathemtical Monthly*, January, 1929.

The Teacher's Department

Edited by
JOSEPH SEIDLIN

INTERESTING THE SUPERIOR STUDENT*

By E. R. SLEIGHT
Albion College, Albion, Michigan

Due to the fact that students from Albion College have appeared rather regularly on these programs, several of you have expressed a desire that I say something about the methods used to interest the undergraduates in research problems. And this is the reason why I am here. In attempting to put into words some of the things we do, I have avoided the personal as much as possible, but in spite of this fact, this paper may have some of the ear marks of an autobiography. However, I am sure you will understand that the purpose involved is an attempt to explain our procedure, which covers better than a quarter of a century as head of the department of Mathematics at Albion College.

The student of unusual ability has always interested me. Even in the days of my experience teaching in preparatory schools such a student was a distinct challenge, and I made it a part of my teaching program to prepare something out of the ordinary for him so that the class hour would not be an hour of marking time, and listening to uninteresting explanations. To what extent should this spirit of research among the undergraduates be encouraged? President Angell of Yale, in one of his lectures, made this statement, "Individual initiative, resourceful ingenuity, imagination, vision must be kept at high pitch." The modern spirit of scientific research has gripped us, and I am not sure but that it has become something of a dominant note even in our teaching. In fact the development of personal initiative on the part of the student seems to be the question uppermost on the part of faculties in many of our more progressive undergraduate schools.

In an article which appeared early in 1926 in School and Society, Dr. G. R. Williams of Amherst College emphasized the advantages of a plan of teaching that would in some way challenge the superior student. We all know that much of the college program is geared to the average student, and much of the extra time is devoted to those of

*Read at Ann Arbor at meeting of Michigan section of the Mathematical Association of America, March 19, 1935.

the lower rank. Perhaps this is not true in the large universities, but I believe it to be true more than ever in many of the smaller institutions. The effort to obtain and retain students is causing undue emphasis on behalf of the student who is having his scholastic difficulties. Examine the personnel program in many institutions and it seems to be an effort to aid the freshman who is unable to gear into the college program. However, here and there we seem to be finding a new note, and the student with an inquisitive mind is getting some attention. Even yet most institutions lay down certain requirements which must be fulfilled by each and every student before he can receive a degree from that institution. In this way initiative on the part of a student is effectually stifled. A few years ago a certain young man was dropped from college because of poor scholarship. A few days later it was announced from the commencement platform that this same young man had been granted a prize for having turned in the most original theme in English composition that had been presented by any freshman during the college year.

Then, too, the members of the faculties in most institutions are experiencing difficulties these days in finding time to devote to the fields of personal interests. Lack of funds, a heavy teaching load, and numerous outside duties take the toll of both time and strength. But I feel sure that we can make it possible to give to those students of unusual ability and inquiring minds some time and a bit of inspiration to help them with their problems. To discover such students and to awaken interest in them is our first duty. To do this some mechanical means may be employed. The methods which we have used have been rather satisfactory, but others may have other methods, and still arrive at the same, or even better results. In the early stages of a student's college career, necessarily he will be solving problems that have already been solved. But how to challenge the superior student so that he will be interested in something that in no way challenges the majority of the class, is the question. In assigning a lesson certain problems may be suggested, stating frankly that these are not a part of the lesson, but that it is hoped that someone will have sufficient interest to present solutions. Have such solutions explained to the rest of the class. It is rather surprising how many will become interested after a few such problems have been explained. Then at the time of tests, one or two rather difficult questions may be asked, with the understanding that these are not a part of the test, and are to be solved only by those who have completed the required work. These may be designated as premium problems, to be used either to increase the grade, or allowed to accumulate and eventually substituted for a test.

In every course there are many problems of this character that suggest themselves. In trigonometry, the summation of such series as $\cos\theta$, $\cos 2\theta$, $\cos 3\theta$, etc., have been found useful. Most texts in analytical geometry pay very little attention to the subject of determinants, and their many useful applications in this field. A study of such applications becomes challenging especially to the student who is interested in pure mathematics. In calculus, I find that such problems as harmonizing the exponential definitions of hyperbolic functions with the geometric definitions have been very effective.

Some other influences have had a decided effect in awakening interest among the students. Our Mathematics Club composed of juniors and seniors who are majoring in the department, and who have attained an average of at least B in the courses taken in this field previous to election into the club, has been a great help. Any student doing any sort of original work has an opportunity to present his results to the club. To indicate the influence of this organization in the 23 years of its existence, we recently made a survey of its membership. During these years, 208 students have been elected to membership, fourteen of whom are not in college. Of the remaining 194, sixteen have entered the field of engineering, 46 have done graduate work to the extent of receiving the Master's degree, while fourteen have been granted the Doctor's degree. Two of the members deserted us—one entered the medical profession, and the other became a lawyer. In addition to those already enumerated, six others are now doing graduate work, some of whom are hoping soon to fulfill the requirements for the doctorate.

Some years ago we introduced honors courses for seniors. These have also had some influence in maintaining interest in mathematics. In these courses a student is given an outline of material, and he does all the reading and investigating by himself. He meets the instructor weekly for reports and discussion. On the other hand, no one thing has stirred up more interest than the courtesy of the program committee of this association. To appear on this program is the ultimate goal of every student in Albion College who is able to do anything in an original way in mathematics.

My first thrill came several years ago when a student laid on my desk a thorough investigation of the probelm concerning the number and types of normals that may be drawn to a parabola. The inspiration came to him as a result of a class discussion. The author had made the statement that from any point in the plane of a parabola three normals may be drawn to the curve. Some of the students took exception to the statement, showing clearly that from some

points only one could be drawn, forgetting that the general statement included the possibility of imaginary normals. It was readily found that the solution of the problem depended upon an equation of the third degree, and in discussing this equation, the young man in question applied all of his knowledge of Theory of Equations, as well as the theory of envelopes. But he was not satisfied with investigating the parabola, but also obtained some very interesting results concerning normals to the central conics. I then realized the possibilities of undergraduate research, and the challenge has become more and more interesting.

Two years ago we had one of the best junior groups that it has been my privilege to teach. Many interesting questions arose from class discussions. Several of the students brought in unusual curves that could be used for trisecting the angle.

That same year from a class in higher algebra came two very worthwhile pieces of work—one on the divisibility of numbers and one on series summation. After studying the remainder theorem and the proofs of divisibility of numbers by use of congruencies, one of the young men decided that this whole subject* might well be discussed by use of the remainder theorem. Another young man conceived the notion that it would be possible to assign a certain definite value as the limit of the sum of n terms of a converging power series. He reduced the problem to an nth degree equation containing only three terms, the nth degree term, the first degree term and a constant. But in attempting to solve this equation as n approached infinity, he found trouble. Out of his investigations, however, came two very interesting formulas for summing series. Both of these young men have appeared on the Association programs.

We are always glad to receive problems that are within the scope of an undergraduate. Only recently one of our alumni, who is now associate professor of physics in a state university, wrote me that he was working on a problem concerning the behavior of a magnetic field in the neighborhood of a solenoid. The problem reduced itself to the possibility of summing an infinite number of series. For some reason the departments of mathematics and physics in this university had failed to recognize that these series are modified forms of the expansion of $(1-x)^{-1/3}$. the sum of the nth series being represented by $1+x+x^2+x^3 \ldots\ldots$ I suggested this problem to one of my juniors, and he came back next morning with it completely solved.

Just recently this same young man solved another problem which had been presented to the Mathematics Department for solution.

*Appeared in Mathematics News Letter, January, 1934.

The interesting part to me is that he solved it, but due to the fact that the solution involves a single equation with four unknowns, and under very restricted conditions—the thing that did interest me was the original methods which he used, not having had the theory of the solution of indeterminate equations, and never having heard of restricted equations. But it is not problems of this type for which we are looking. We want something that requires a sustained effort—but yet problems that do not lead into fields too far remote from undergraduate preparation. After all the type of exercise that most interests a student is one that he has discovered for himself. If we can but develop the research spirit in him so that he is on the lookout for interesting problems, I believe we have reached the goal toward which we have aimed. One of the difficulties arises, however, when a student gets started on an investigation that takes him into depths for which he is not prepared. At such times I find that I am compelled to step lively to keep up—a condition rather desirable if one hopes to keep from stagnation in an undergraduate institution.

Perfect frankness is necessary in dealing with the superior student. He must be permitted to express himself freely and unhampered, even at times when it becomes embarrassing for the instructor. Just recently in one of my classes I had made rather a lengthy explanation, when one of the boys remarked, "Dr. Sleight, I don't believe that", and proceeded to prove that I was wrong.

To me the greatest satisfaction comes from the inspiration which I have received from these young people. Again referring to President Angell's statement that all education should be so geared that "individual initiative, resourcefulness, ingenuity and imagination must be kept at high pitch", I am wondering if this aim cannot best be obtained by creating a spirit of research even among our undergraduates. In this way we have brought the student face to face with the need of thinking a problem through for himself, and of devising methods for its solution. Perhaps this is the ultimate aim of education, and if so what better method can be devised?

Notes and News

Edited by
L. J. ADAMS and I. MAIZLISH

Meetings of the American Mathematical Society are scheduled as follows:

1. Lexington, Kentucky, November 29-30, 1935.

By invitation of the Program Committee, addresses will be delivered by Professor L. L. Dines on *Convex Domains and Linear Inequalities*, and by Professor C. D. Latimer on *The Arithmetic of Generalized Quaternions*.

2. California Institute of Technology, Pasadena, November 30, 1935.
3. St. Louis, Missouri, Annual Meeting, December 31, 1935, January 3, 1936.

By invitation of the Program Committee, addresses will be delivered by Professors Szegö and Synge.

A total of ninety-one doctorates with mathematics as major subject were conferred by the universities of the United States and Canada during the year 1934. In addition, three doctorates with mathematical physics as major subject were awarded.

In 1934 the following volumes were published as the colloquium series of the American Mathematical Society:

1. *Lectures on Matrices.* J. H. M. Wedderburn.
2. *The Calculus of Variations in the Large.* Marston Morse.
3. *Fourier Transforms in the Complex Domain.* R. E. A. C. Paley and Norbert Wiener.

Abhandlungen aus dem Mathematischen Seminar der Hamburgischen Universität is a collection of research papers which appears annually in one volume of four numbers. Emphasis is given to research in higher algebra, arithmetic and geometry produced by fellows of the mathematical seminary at Hamburg. All articles are in German.

The National Council of Teachers of Mathematics will meet at the Coronado Hotel, St. Louis, Missouri, December 31, 1935-January 1, 1936 in conjunction with the Mathematical Association of America and Section A of the A. A. A. S. The meetings will include addresses on topics of interest to secondary school teachers of mathematics.

The Mathematical Association of England has elected as honorary members E. Borel of the University of the Sorbonne, G. H. Hardy of the University of Cambridge, J. Hadamard of the Collège of France, D. E. Smith of Columbia University (emeritus), and E. T. Whitaker of the University of Edinburgh.

Professor Burton H. Camp recently addressed a letter to three hundred members of the American Statistical Association asking for suggestions as to statistical problems which in their opinion justified further mathematical research. The summary of the answers appears in the March, 1935 supplement of the Journal of the American Statistical Association. The reader is referred to this summary, where the suggested fields for research are listed. They include references to time series, trends, demand curves and the theory of small samples, among others. Mathematicians interested in statistics would probably be stimulated by this list of specific topics which seem to require more careful and complete treatment than has so far been given.

It is quite a common practice in some of the foreign journals to publish as a frontispiece the photograph of a recently deceased mathematician. The portrait is followed by a rather complete biographical note. Students of the history of mathematics will find much of value in these biographies.

The Annual Convention of Central Association of Science and Mathematics Teachers will be held in Chicago at the Palmer House on November 29 and 30, 1935. The list of speakers includes Dr. G. D. Birkhoff of Harvard and Professor Walter W. Hart of the University of Wisconsin.

The Tohoku Mathematical Journal, published by the Tohoku Imperial University at Sendai, Japan, has become a favorite outlet for American research workers. Some numbers of this journal contain as many as four hundred pages, of which almost half is comprised of American contributions. Almost all of the articles are in English and they include many noteworthy discoveries. Only the last twenty to fifty pages are in the Japanese language.

The Final Contest of the Southern Intercollegiate Mathematics Association was held at Centenary College, Shreveport, La., on May 11th, 1935. Millsaps College, McMurry College, the College of the Ozarks, and Centenary College competed. Centenary College won the contest and was awarded the Association's Cup for temporary possession. The Final Contest for the 1935-36 session will be held at the Louisiana State University.

Problem Department

Edited by
T. A. BICKERSTAFF

This department aims to provide problems of varying degrees of difficulty which will interest anyone who is engaged in the study of mathematics.

All readers, whether subscribers or not, are invited to propose problems and to solve problems here proposed.

Problems and solutions will be credited to their authors.

While it is our aim to publish problems of most interest to the readers, it is believed that regular text-book problems are, as a rule, less interesting than others. Therefore, other problems will be given preference when the space for problems is limited.

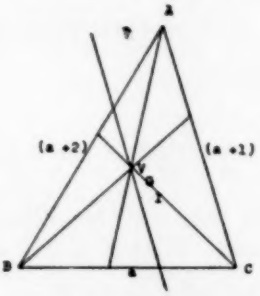
Send all communications about problems to T. A Bickerstaff, University, Mississippi.

SOLUTIONS

No. 90. *Proposed by Walter B. Clarke*, San Jose, Cal.

Show that if the sides of a triangle are equivalent to consecutive integers, the line joining the incenter and verbicenter is parallel to one of the sides. (Verbicenter is the concurrent point of lines joining each vertex to the point half way around the perimeter from the vertex.)

Solved by Karleton W. Crain, Purdue University.



Let ABC be the triangle whose sides are a, (a+1), and (a+2) respectively. Using this as the triangle of reference, the trilinear coordinates of the incenter, I, are proportional to (1:1:1) and since

the centroid, G(sin B sin C, sin C sin A, sin A sin B), is on the line VI (cf. Johnson, R. A., Modern Geometry, pages 149, 225.), the equation of VGI is –

$$\begin{vmatrix} \alpha, & \beta, & \gamma \\ \sin B \sin C, & \sin C \sin A, & \sin A \sin B \\ 1, & 1, & 1 \end{vmatrix} = 0$$

or

$$\begin{aligned} &\alpha(\sin C \sin A - \sin A \sin B) \\ &+\beta(\sin A \sin B - \sin B \sin C) \\ &+\gamma(\sin B \sin C - \sin C \sin A) = 0. \end{aligned}$$

The equation of CA is $\beta = 0$. The condition that these lines be parallel is that they intersect on the line at infinity, whose equation for this special triangle of reference is, $a\alpha + (a+1)\beta + (a+2)\gamma = 0$. Therefore, the following determinant must vanish,

$$\begin{vmatrix} 0, & 1, & 0 \\ a, & (a+1), & (a+2) \\ (\sin C \sin A - \sin A \sin B, & (\sin A \sin B - \sin B \sin C), & (\sin B \sin C - \sin C \sin A) \end{vmatrix}$$

Since the sides of a triangle are proportional to the sines of their corresponding angles, we may substitute them in the third row of the above determinant and we have,

$$\begin{vmatrix} 0, & 1, & 0 \\ a, & (a+1), & (a+2) \\ a, & -2(a+1), & (a+2) \end{vmatrix}$$

which is identically zero. Therefore the line connecting the verbicenter with the incenter is parallel to CA.

Also solved by J. Rosenbaum, Hartford, Conn.; A. C. Briggs, Wilmington, Ohio, and the proposer.

No. 91. *Proposed by W. V. Parker*, Georgia, Tech.

P is a fixed point of the ellipse

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$$

and X and Y are two other points of the ellipse. Prove that the area of the triangle PXY is greatest when X and Y are also on the ellipse

$$\frac{(x+h)^2}{a^2} + \frac{(x+k)^2}{b^2} = 1,$$

where (h, k) are the coordinates of P.

Solution by J. Rosenbaum, Hartford, Conn.

If P′Q′ is a diameter of the circle C_1 (solution to 89) then since P′X′Y′ is equilateral the circle C_1 with Q′ as center and radius equal to a, will pass through X′ and Y′. Hence the projection of C_2 on the xy plane will pass through X and Y, and it is easily seen that this projection of C_1 is the second ellipse of this problem.

Also solved by A. C. Briggs and the proposer.

No. 95. *Proposed by Edmund N. Yeager*, Toledo (Ohio) Teachers College.

A man enters a bank and has a check cashed. The teller mistakes the figures and pays cents for dollars and dollars for cents. The man then pays a bill for \$24.11 after which he finds he has twice as much money as the face of the original check. What was the face of the check. Show that the solution is unique.

Solved by David Blackwell, University of Illinois.

Let x = dollars the check called for.
y = cents the check called for.

Then $$100\,y+x-2411=200\,x+2y$$

or $$98\,y-199\,x=2411$$

which is a simple linear equation in two unknowns. Solving by means of continued division, it is found that

$$x=13-98\,z$$

$$y=51-199\,z$$

where z may assume any value. By the conditions of the problem, x and y must be positive integers less than 100. It will be seen that $z=0$ is the only value of z yielding values of x and y which meet these conditions. Therefore the check called for \$13.51.

Also solved by A. C. Briggs, Wilmington, Ohio; Hobson M. Zerbe, Wilkes-Barre, Penn., M. L. Constable, Philadelphia, and Jerome Maxwell, University of Mississippi.

PROBLEMS FOR SOLUTION

Note: Attention is hereby directed to the following numbered problems which have appeared in this department but for which no solutions have been submitted: Numbers 87, 82, 80, 81, 63, 62, 58, 48, 45, 30, 28, 24 and 23.

LATE SOLUTIONS

No. 93, by A. C. Briggs.

Book Reviews

Edited by
P. K. SMITH

Mathematics and the question of the cosmic mind with other essays. By Cassius Jackson Keyser, Adrian Professor Emeritus of Mathematics, Columbia University, New York, 1935, pp. v+121. Price 75 cents.

Mathematics is unbounded, undivided, and too often unappreciated. The statements in the mathematical literature can be divided into the following three categories: 1) Those which can be proved satisfactorily, 2) Those which can be definitely disproved, and 3) Those which can at present be neither proved nor disproved. The first of these categories includes the greater part of the modern publications and is now growing very rapidly. Recently a well known Belgian mathematician, M. Lecat, published a small volume on the second of these categories under the title *Erreurs de Mathématiciens des origines à nos jours*, 1935, in which he cited errors committed by such well known writers as N. H. Abel, A. L. Cauchy, A. Cayley, K. F. Gauss, J. J. Sylvester and many others, with references to where they were corrected.

Many of the statements of the present volume belong to the third of the three categories noted above, which includes many of the most inspiring statements of our literature and has been enriched during recent years by various other writings by the same author. The number and the frequency of these publications prove that many mathematicians enjoy, at least occasionally, to enter these borderlands of our subject, not only to refresh themselves but also to secure new points of view and a broader outlook. It should not be inferred that these borderlands furnish richer inspiration to those who have really mastered the elements of their subjects than the actual domain of mathematics. In fact, the contrary is true, but they also furnish inspiration and the variety is refreshing.

The mathematician deals continually with enrapturing ideas which become more inspiring as he enters more deeply into his special field of study. On this account he is too much inclined to bury himself in ideas which only a few of his associates can appreciate. Such borderlands as those represented by the volume under review enlarge his sympathetic contacts and hence they may be entered at least

occasionally, with profit. This volume consists of reprints with some alterations, of articles that have recently appeared in *The American Scholar*, the *Scripta Mathematica*, and the *Yale Law Journal*, under the following six titles: "The meaning of mathematics, the bearings of mathematics, mathematics and the question of cosmic mind, mitigating the tragedy of our modern culture, on the study of legal science, and William Benjamin Smith."

These headings indicate that a variety of somewhat disconnected subjects are discussed in this little volume, which was prepared for the *Scripta Mathematica Library* and constitutes its second volume, the first being entitled *Poetry of Mathematics and other essays* by David Eugene Smith. The object of this library is to provide mathematical books for non-mathematicians. It is to be hoped that the series will tend towards winning many new friends for mathematics who might be deterred from entering this field by the more technical mathematical literature. One way of increasing the influence of mathematics is to make it interesting to a wider circle of those enjoying intellectual activities and this way seems to be the objective of this series. As an instance of the type of utterances in the present volume we quote the following from page 62: "Mind is the sole Reality, all else—the external universe—being only a mind-required, mind-made, mind-sustained, mind-symbolizing, mind-revealing Pageant."

G. A. MILLER.
University of Illinois.

Differential Equations for Electrical Engineers. By Philip Franklin, New York, John Wiley & Sons, 1933. 299+vii pages.

This book is the result of a course given to junior electrical engineers at Massachusetts Institute of Technology over a period of years. It is designed to follow ordinary differential equations.

Chapters I and II are introductory in nature: the first chapter deals with complex numbers—it contains applications to alternating current theory (with stress on the notation used by the electrical engineer) and two articles on conformal representation; the second chapter is devoted to mean values and Fourier series.

Chapters III to VI are concerned with the theory of differential equations. In the third chapter the theory of linear ordinary differential equations with constant coefficients and of systems of such equations is given. Applications are made to electric circuits after the introduction of the concept of impedance. In the fourth chapter we meet with partial derivative and partial differential equations

solvable by direct integration, by the method of Lagrange (linear first order), or by the classical devices associated with linear homogeneous equations with constant coefficients. The next two chapters give a derivative of several partial differential equations of mathematical physics, including the flow of electricity in a cable, heat flow, fluid motion, and wave motion, as well as problems involving initial and boundary conditions.

The last two chapters give proofs concerning analytic functions and Fourier series omitted in the introductory chapters; they may serve as reference material or as an introduction to more advanced treatments.

There are a few obvious errors in printing. Some objections could be raised to indicating the measure of an angle by so many radians plus so many degrees as on page 20. It is also tacitly assumed on page 20 that in the expression $e^{iwt}w$ is positive. The problems at the end of Chapter I might mean more to the student if references were given to elementary texts in which the physical theory underlying the applications is developed or if problems of less advanced character were chosen. Some interesting problems concerning the variation of current or voltage when the impedance is varied would have been quite appropriate at this point.

On the whole the book is a welcome arrival in the circle of differential equation texts. All too long has the mathematical passage from the differential equations of circuit theory to the complex numbers used by the electrical engineer been passed over as if it were something mysterious by the writers of elementary electrical engineering texts. Here the issue is met squarely and in addition a glimpse of many other fields of application is given to the future engineer.

W. E. BYRNE.

Toward the close of the nineteenth century numerous disturbing paradoxes appeared in mathematical analysis. In the subsequent critical examination of the foundations of mathematics further paradoxes sprang up like weeds in the very logic which had been used in all confidence in some of the most vital parts of analysis. The following humorous translation of one of these paradoxes (Russell's) into ordinary language may amuse the reader provided it does not cause him to lose his mind; the translation also is Russell's: A male barber shaves all those men in town who do not shave themselves, and he shaves no other men. Does he shave himself? Both answers are easily seen to be wrong.—Eric T. Bell in *"Current History."*